

```
Clear[initialcrossing, initialmatrix, braidword, Quad, A, F, HigherOrder,
kerbeforesub, keraftersub, exp2, poly, poly2, tr, knot, n, p1, p2, p3, p4]
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knot = {9, 14};
f[{{i_, n_}}] := If[n > 0, ConstantArray[{i, i + 1}, {n}], ConstantArray[{i + 1, i}, {-n}]]
list = Flatten[f /@ KnotData[knot, "BraidWord"]];
initialcrossing = firstcrossing[Take[list, {1, 2}]];
initialmatrix = firstmatrix[Take[list, {1, 2}]];
braidword = Sequence[Partition[Take[list, 2 - Length[list]], 2]];
n = KnotData[knot, "BraidIndex"];
firstcrossing[{{x_, y_}}] := If[x < y, HOx,y, INVy,x];
firstmatrix[{{x_, y_}}] := If[x < y, Matrixx,y, Imatrixy,x];
alex = KnotData[knot, "AlexanderPolynomial"] [t^2];
amp = KnotData[knot, "Amphichiral"];
rep2[i_, j_] :=
Which[i == j == 0, 1, True, ((i! j! Coefficient[exp2, d2^i z2^j]) /. {z2 → 0, d2 → 0})]
rep3[i_, j_, k_, l_] := Which[i == j == k == l == 0, 1, True,
((i! j! k! l! Coefficient[exp2, d2^i z2^j d3^k z3^l]) /. {z2 → 0, d2 → 0, z3 → 0, d3 → 0})]
rep4[i_, j_, k_, l_, m_, n_] := Which[i == j == k == l == m == n == 0, 1, True,
((i! j! k! l! m! n! Coefficient[exp2, d2^i z2^j d3^k z3^l d4^m z4^n]) /.
{z2 → 0, d2 → 0, z3 → 0, d3 → 0, z4 → 0, d4 → 0})]
rep5[i_, j_, k_, l_, m_, n_, o_, p_] := Which[i == j == k == l == m == n == o == p == 0, 1, True,
((i! j! k! l! m! n! o! p! Coefficient[exp2, d2^i z2^j d3^k z3^l d4^m z4^n d5^o z5^p]) /.
{z2 → 0, d2 → 0, z3 → 0, d3 → 0, z4 → 0, d4 → 0, z5 → 0, d5 → 0})]
rep6[i_, j_, k_, l_, m_, n_, o_, p_, q_, r_] :=
Which[i == j == k == l == m == n == o == p == q == r == 0, 1, True, ((i! j! k! l! m! n! o! p! q!
r! Coefficient[exp2, d2^i z2^j d3^k z3^l d4^m z4^n d5^o z5^p d6^q z6^r]) /.
{z2 → 0, d2 → 0, z3 → 0, d3 → 0, z4 → 0, d4 → 0, z5 → 0, d5 → 0, z6 → 0, d6 → 0})]
trace2[s_] := Expand[Plus@@(s /. Rule[{a_, b_}, c_] => rep2[a, b] c)]
trace3[s_] := Expand[Plus@@(s /. Rule[{a_, b_, e_, f_}, c_] => rep3[a, b, e, f] c)]
trace4[s_] :=
Expand[Plus@@(s /. Rule[{a_, b_, e_, f_, g_, h_}, c_] => rep4[a, b, e, f, g, h] c)]
trace5[s_] := Expand[
Plus@@(s /. Rule[{a_, b_, e_, f_, g_, h_, i_, j_}, c_] => rep5[a, b, e, f, g, h, i, j] c)]
trace6[s_] := Expand[Plus@@(s /. Rule[{a_, b_, e_, f_, g_, h_, i_, j_, k_, l_}, c_] =>
rep6[a, b, e, f, g, h, i, j, k, l] c)]
qh[n_] := (1 + h Coefficient[Product[1 + 2 h zi di, {i, 2, n}], h, 1])
pL,k[i_, j_] :=
Which[i == k && j == L, t, i == L && j == k, t, i == j == L, 0, i == j == k, 1 - t^2, i == j, 1, True, 0];
MatrixL,k := Array[pL,k, {n, n}];
ImatrixL,k := Inverse[MatrixL,k];
SZL,k[zj] := Expand[Sum[zi ImatrixL,k[[i, j]], {i, 1, n}]];
SderL,k[dj] := Expand[Sum[MatrixL,k[[i, j]] di, {i, 1, n}]];
ISZL,k[zj] := Expand[Sum[zi MatrixL,k[[i, j]], {i, 1, n}]];
ISderL,k[dj] := Expand[Sum[ImatrixL,k[[i, j]] di, {i, 1, n}]];
Higher[a_, b_, w_, x_] :=
1 + h  $\left( 2 a b w x - \frac{2 a^2 w x}{t} + 2 a^2 t w x + 3 a^2 x^2 - \frac{a^2 x^2}{t^2} + \frac{a b x^2}{t} - 3 a b t x^2 - 2 a^2 t^2 x^2 \right)$ 
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$$\text{Inv}[a_ , b_ , w_ , x_] := 1 + h \left(b^2 * w^2 - t^2 b^2 * w^2 + \frac{a * b * w^2}{t} + t a * b * w^2 - 2 a * b * w * x \right)$$

HO_{i_,j_} := Higher[z_i, z_j, d_i, d_j]

INV_{i_,j_} := Inv[z_i, z_j, d_i, d_j]

Der_i[f_] := f + Sum[(1/k!) D[f, {w_i, k}, {x_i, k}], {k, 1, 4}]

NO3[L_, {i_, j_}] /; i < j :=

(Der_j[Der_i[Expand[1 + h Coefficient[(L /. Flatten[Table[{z_k → Sz_{i,j}[z_k],
d_k → Sder_{i,j}[d_k]], {k, 1, n}]]] /. {d_i → w_i, d_j → w_j}] *
Higher[x_i, x_j, d_i, d_j, h, 1]]]]] /. {w_i → d_i, w_j → d_j, x_i → z_i, x_j → z_j}

NO3[L_, {i_, j_}] /; i > j :=

(Der_j[Der_i[Expand[1 + h Coefficient[(L /. Flatten[Table[{z_k → ISz_{j,i}[z_k],
d_k → ISder_{j,i}[d_k]], {k, 1, n}]]] /. {d_i → w_i, d_j → w_j}] *
Inv[x_j, x_i, d_j, d_i, h, 1]]]]] /. {w_i → d_i, w_j → d_j, x_i → z_i, x_j → z_j}

subd = Flatten[Table[d_i → w_i, {i, 2, n}]];

subz = Flatten[Table[z_i → x_i, {i, 2, n}]];

g[L_, {x_, y_}] := If[x < y, L.(Matrix_{x,y}), L.(Imatrix_{y,x})]

qhend3[f_] := Expand[(f /. subd) + h (Coefficient[qh[n], h, 1] /. subz)]

Quad = Expand[Fold[g, initialmatrix, braidword]];

a_{i_,j_} := If[i == j, Quad[[i, j]] - 1, Quad[[i, j]]];

A = Table[a_{i,j}, {i, 2, n}, {j, 2, n}];

F = Inverse[-A];

initialcrossing

$$1 + h \left(-\frac{2 d_1 d_2 z_1^2}{t} + 2 t d_1 d_2 z_1^2 + 3 d_2^2 z_1^2 - \frac{d_2^2 z_1^2}{t^2} - 2 t^2 d_2^2 z_1^2 + 2 d_1 d_2 z_1 z_2 + \frac{d_2^2 z_1 z_2}{t} - 3 t d_2^2 z_1 z_2 \right)$$

AbsoluteTiming[HigherOrder3 = Fold[NO3, initialcrossing, braidword];]

{3.08242, Null}

AbsoluteTiming[kerbeforesub3 = ((1 = qhend3[HigherOrder3];

Do[1 = Der_i[1], {i, 2, n, 1}];

1) /. Flatten[Table[{x_i → z_i, w_i → d_i}, {i, 2, n}]]];]

{0.794898, Null}

sub = Flatten[Table[z_k → Expand[Sum[Quad[[i, k]] z_i, {i, 1, n}]], {k, 1, n}]]

$$\left\{ \begin{aligned} z_1 &\rightarrow -z_1 + t^2 z_1 - \frac{2 z_2}{t} + 3 t z_2 - t^3 z_2 + \\ &3 z_3 + \frac{z_3}{t^4} - \frac{2 z_3}{t^2} - t^2 z_3 - \frac{z_4}{t} + 2 t z_4 - t^3 z_4 + 3 z_5 - \frac{z_5}{t^2} - 3 t^2 z_5 + t^4 z_5, \\ z_2 &\rightarrow \frac{z_1}{t} - 5 t z_1 + 8 t^3 z_1 - 5 t^5 z_1 + t^7 z_1 - 7 z_2 + \frac{z_2}{t^2} + 14 t^2 z_2 - 13 t^4 z_2 + 6 t^6 z_2 - \\ &t^8 z_2 - \frac{z_3}{t^5} + \frac{5 z_3}{t^3} - \frac{12 z_3}{t} + 17 t z_3 - 14 t^3 z_3 + 6 t^5 z_3 - t^7 z_3 - 5 z_4 + \frac{z_4}{t^2} + 10 t^2 z_4 - \\ &10 t^4 z_4 + 5 t^6 z_4 - t^8 z_4 + \frac{z_5}{t^3} - \frac{6 z_5}{t} + 16 t z_5 - 20 t^3 z_5 + 15 t^5 z_5 - 6 t^7 z_5 + t^9 z_5, \\ z_3 &\rightarrow -3 z_1 + \frac{z_1}{t^2} + 5 t^2 z_1 - 4 t^4 z_1 + t^6 z_1 + \frac{z_2}{t^3} - \frac{4 z_2}{t} + 9 t z_2 - 9 t^3 z_2 + 5 t^5 z_2 - \\ &t^7 z_2 + 11 z_3 - \frac{z_3}{t^6} + \frac{4 z_3}{t^4} - \frac{8 z_3}{t^2} - 10 t^2 z_3 + 5 t^4 z_3 - t^6 z_3 + \frac{z_4}{t^3} - \frac{4 z_4}{t} + 7 t z_4 - \\ &7 t^3 z_4 + 4 t^5 z_4 - t^7 z_4 + 11 z_5 + \frac{z_5}{t^4} - \frac{5 z_5}{t^2} - 14 t^2 z_5 + 11 t^4 z_5 - 5 t^6 z_5 + t^8 z_5, \\ z_4 &\rightarrow \frac{z_1}{t^3} - \frac{3 z_1}{t} + 3 t z_1 + 2 t^3 z_1 - 5 t^5 z_1 + 4 t^7 z_1 - t^9 z_1 + 6 z_2 + \frac{z_2}{t^4} - \frac{4 z_2}{t^2} - 8 t^4 z_2 + \\ &9 t^6 z_2 - 5 t^8 z_2 + t^{10} z_2 - \frac{z_3}{t^7} + \frac{4 z_3}{t^5} - \frac{7 z_3}{t^3} + \frac{5 z_3}{t} + 3 t z_3 - 10 t^3 z_3 + 10 t^5 z_3 - \\ &5 t^7 z_3 + t^9 z_3 + 6 z_4 + \frac{z_4}{t^4} - \frac{4 z_4}{t^2} - 2 t^2 z_4 - 5 t^4 z_4 + 7 t^6 z_4 - 4 t^8 z_4 + t^{10} z_4 + \\ &\frac{z_5}{t^5} - \frac{5 z_5}{t^3} + \frac{10 z_5}{t} - 7 t z_5 - 4 t^3 z_5 + 12 t^5 z_5 - 11 t^7 z_5 + 5 t^9 z_5 - t^{11} z_5, \\ z_5 &\rightarrow 2 z_1 - \frac{z_1}{t^2} - t^2 z_1 - \frac{z_2}{t^3} + \frac{3 z_2}{t} - 3 t z_2 + t^3 z_2 - 3 z_3 + \frac{z_3}{t^6} - \frac{3 z_3}{t^4} + \frac{4 z_3}{t^2} + \\ &t^2 z_3 - \frac{z_4}{t^3} + \frac{3 z_4}{t} - 2 t z_4 + t^3 z_4 - 5 z_5 - \frac{z_5}{t^4} + \frac{4 z_5}{t^2} + 3 t^2 z_5 - t^4 z_5 \} \end{aligned} \right.$$

AbsoluteTiming[keraftersub = kerbeforesub3 /. sub;]

{0.043467, Null}

sub3 = Simplify[Collect[

Flatten[Table[{z_i → z_i + Expand[Sum[F[[k, i - 1]] Quad[[1, k + 1]] z₁, {k, 1, n - 1}]],
d_i → d_i + Expand[Sum[F[[i - 1, 1]] Quad[[1 + 1, 1]] d₁, {1, 1, n - 1}]],
{i, 2, n}]], {z₁, z₂, z₃, z₄, z₅, d₁, d₂, d₃, d₄, d₅}]]

$$\left\{ \begin{aligned} z_2 &\rightarrow \frac{z_1}{t} + z_2, d_2 \rightarrow \frac{d_1}{t} + d_2, z_3 \rightarrow \frac{z_1}{t^2} + z_3, \\ d_3 &\rightarrow \frac{d_1}{t^2} + d_3, z_4 \rightarrow \frac{z_1}{t^3} + z_4, d_4 \rightarrow \frac{d_1}{t^3} + d_4, z_5 \rightarrow \frac{z_1}{t^4} + z_5, d_5 \rightarrow \frac{d_1}{t^4} + d_5 \} \end{aligned} \right.$$

AbsoluteTiming[twopoly = keraftersub /. sub3]

{1.00276,

$$1 + \dots 7215 \dots + 96 h t^8 \left(\dots 1 \dots \right)^2 \left(\dots 1 \dots \right)^2 - 8 h t^{10} \left(\frac{d_1}{t^4} + d_5 \right)^2 \left(\frac{z_1}{t^3} - \frac{3 z_1}{t} + 3 t z_1 + 2 t^3 z_1 - 5 t^5 z_1 + \dots 46 \dots + 12 t^5 \left(\frac{z_1}{t^4} + z_5 \right) - 11 t^7 \left(\frac{z_1}{t^4} + z_5 \right) + 5 t^9 \left(\frac{z_1}{t^4} + z_5 \right) - t^{11} \left(\frac{z_1}{t^4} + z_5 \right)^2 \right\}$$

large output | show less | show more | show all | set size limit...

sub4 = Sum[(Expand[Sum[F[[i - 1, j - 1]] d_i z_j, {j, 2, n}, {i, 2, n}]]) ^i / i!, {i, 1, 4}]

$$\frac{4 d_2 z_2}{2 t^2 - 9 t^4 + 15 t^6 - 9 t^8 + 2 t^{10}} - \frac{18 t^2 d_2 z_2}{2 t^2 - 9 t^4 + 15 t^6 - 9 t^8 + 2 t^{10}} + \frac{32 t^4 d_2 z_2}{2 t^2 - 9 t^4 + 15 t^6 - 9 t^8 + 2 t^{10}} - \frac{26 t^6 d_2 z_2}{\dots 6 \dots + 2 t^{10}} + \dots 153 \dots + \frac{2 \dots 2 \dots z_5}{\dots 1 \dots} + \frac{1}{2} \left(\dots 1 \dots \right)^2 + \frac{1}{6} \left(\dots 159 \dots + \frac{\dots 1 \dots}{\dots 1 \dots} \right)^3 + \frac{1}{24} \left(\dots 159 \dots + \frac{2 t^{10} d_5 z_5}{2 t^2 - 9 t^4 + \dots 1 \dots - 9 \dots 1 \dots + 2 t^{10}} \right)^4$$

large output | show less | show more | show all | set size limit...

AbsoluteTiming[exp2 = Expand[1 + sub4];]

{3.94673, Null}

sub5 = Flatten[Table[{d_i, z_i}, {i, 2, n}]]

{d₂, z₂, d₃, z₃, d₄, z₄, d₅, z₅}

AbsoluteTiming[poly2 = CoefficientRules[twopoly, sub5]]

{25.9999, {{2, 2, 0, 0, 0, 0, 0, 0} → -32 h - $\frac{2h}{t^4} + \frac{13h}{t^2} + 3 h t^2 + 144 h t^4 - 341 h t^6 + 408 h t^8 - 292 h t^{10} + 126 h t^{12} - 30 h t^{14} + 3 h t^{16}$,
 $\dots 223 \dots$, {0, 0, 0, 0, 0, 0, 0, 0} → $1 + \frac{2 h d_1 z_1}{t^8} + \frac{2 h d_1 z_1}{t^6} + \frac{2 h d_1 z_1}{t^4} + \frac{2 h d_1 z_1}{t^2}$ }}

large output | show less | show more | show all | set size limit...

AbsoluteTiming[tr = trace_n[poly2];]

{122.246, Null}

test1 = Together[Coefficient[tr, h, 1]]

$$\left(2 \left(-7 + 44 t^2 - 89 t^4 + 15 t^6 + 190 t^8 - 291 t^{10} + 193 t^{12} - 64 t^{14} + 9 t^{16} \right) \right) / \left(2 - 9 t^2 + 15 t^4 - 9 t^6 + 2 t^8 \right)^2$$

p = If[amp == True, Cancel[Denominator[test1] / alex], Cancel[Denominator[test1] / alex^2]]
t⁸

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series = If[amp == True,
  Normal[Series[(((1 + h test1) (1/alex)) /. {t -> t Exp[h]}) (alex) /. {h -> h/2} /.
    {h -> Sqrt[1 + h] - 1/Sqrt[1 + h]}, {h, 0, 1}]],
  Normal[Series[(((1 + h test1) (1/alex)) /. {t -> t Exp[h]}) alex) /. {h -> h/2}, {h, 0, 1}]]]
1 + 
$$\frac{h (1 - 10 t^2 + 52 t^4 - 138 t^6 + 190 t^8 - 138 t^{10} + 52 t^{12} - 10 t^{14} + t^{16})}{(2 - 9 t^2 + 15 t^4 - 9 t^6 + 2 t^8)^2}$$


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p1 = Expand[Numerator[Coefficient[series, h, 1]]/p]
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$$190 + \frac{1}{t^8} - \frac{10}{t^6} + \frac{52}{t^4} - \frac{138}{t^2} - 138 t^2 + 52 t^4 - 10 t^6 + t^8$$

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SessionTime[]
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179.867585
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